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The quasi-bi-Hamiltonian formulation of the Lagrange top

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Abstract

Starting from the tri-Hamiltonian formulation of the Lagrange top (LT) in a six-dimensional phase space, we discuss the possible reductions of the Poisson tensors, the vector field and its Hamiltonian functions on a four-dimensional space. We show that the vector field of the LT possesses, on the reduced phase space, a quasi-bi-Hamiltonian formulation, which provides a set of separation variables for the corresponding Hamilton–Jacobi equation.

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1. Introduction

The classical theory of separation of variables for the Hamilton–Jacobi equation provides the most effective tool for solving the equations of motion of a given Hamiltonian system. In this framework, the main problem is finding an efficient (possibly algorithmic) way to *produce* a set of separation variables. To this end, two new approaches, stemming from soliton theory, have been recently introduced: the 'magic Sklyanin recipe' [1], based on the Lax representation of the equations of the motion, and the bi-Hamiltonian (bH) approach to separation of variables [2–5], based on the bH structures associated with the equations of motion. A remarkable feature of the latter approach is that if the Hamiltonian system admits a *quasi-bi-Hamiltonian* (qbH) formulation, then a set of separation variables can be algorithmically computed [3]; moreover, the qbH property is independent of the coordinate system in which the bH structure is written down.

The aim of this paper is to apply the approach based on the qbH property to the classical Lagrange top (LT); in particular, we show how the (complex) separation variables for LT, introduced in [6] in an algebraic–geometric setting, arise quite naturally as distinguished functions for its tri-Hamiltonian structure.

The starting point of our analysis is the fact that, on a six-dimensional phase space M, the LT vector field X_L admits a tri-Hamiltonian formulation $X_L = P_\alpha \, \mathrm{d}h_\alpha$ (throughout the paper, the index α takes values 0, 1, 2), each one of the three compatible Poisson tensors P_α possessing two independent Casimir functions.

When one tries to eliminate the Casimir functions by fixing their values, one is faced with a typical situation, occurring also for other bH finite-dimensional integrable systems [5, 7, 8]: to each one of the symplectic leaves S_α, one can restrict only the vector field X_L and the corresponding pair (P_α, h_α), but not the entire triple of the Poisson tensors, so the tri-Hamiltonian formulation of X_L is lost under restriction. Nevertheless, using a more general reduction process *à la Marsden–Ratiu*, we will show that the symplectic leaf S_0 of the Poisson tensor P_0 can be endowed with a Poisson–Nijenhuis structure [9, 10] (hence a bH structure) and that X_L can be given a qbH formulation. So, the separability of LT is obtained from its Hamiltonian structures as a natural outcome of the reduction process.

The paper is organized as follows. In section 2 the tri-Hamiltonian structure of LT is briefly reviewed; in section 3 the main properties of the qbH model are discussed with a view to application to the LT. In sections 4 and 5, respectively, the reduction of the Poisson tensors P_α and of the vector field X_L with its Hamiltonian functions are considered; the qbH formulation for X_L is explicitly constructed, together with a solution of the corresponding Hamilton–Jacobi equation. Our results are summarized in section 6, where some potential extensions of this work are pointed out.

2. The multi-Hamiltonian structure of the Lagrange top

A modern formulation of LT can be found in [11, 12]; as usual in this framework, the components of vectors and covectors and the entries of matrices are referred to the comoving frame, whose axes are the principal inertia axes of the top, with fixed point O.

The phase space M of LT is parametrized by the pair $m = (\omega, \gamma)$, where $\omega = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)^{\mathrm{T}}$ are the angular velocity and the vertical unit vector, respectively. The following notation is introduced: μ is the mass of the top, g the acceleration of gravity, $J = \mathrm{diag}(A, A, cA)$ the principal inertia matrix ($c \neq 1$), $G = (0, 0, a)^{\mathrm{T}}$ is the centre of mass; finally, normalizations are chosen such that $\mu a g/A = 1$.

The Euler–Poisson equations are $\mathrm{d}L_O/\mathrm{d}t = M_O$ (change of the angular momentum) and $\mathrm{d}\gamma/\mathrm{d}t = 0$ (invariance of the vertical unit vector); with the above notation and normalizations, these equations take the well-known form

$$\frac{\mathrm{d}m}{\mathrm{d}t} = X_L(m) \qquad X_L(m) = \begin{pmatrix} (1-c)\omega_2\omega_3 - \gamma_2 \\ -(1-c)\omega_3\omega_1 + \gamma_1 \\ 0 \\ \gamma_2\omega_3 - \gamma_3\omega_2 \\ \gamma_3\omega_1 - \gamma_1\omega_3 \\ \gamma_1\omega_2 - \gamma_2\omega_1 \end{pmatrix}. \tag{2.1}$$

The LT vector field X_L can be given a tri-Hamiltonian formulation

$$X_L = P_0 \, \mathrm{d}h_0 = P_1 \, \mathrm{d}h_1 = P_2 \, \mathrm{d}h_2. \tag{2.2}$$

The compatible Poisson tensors P_α, written in block-matrix form, are

$$P_0 = \begin{pmatrix} 0 & B \\ B & C \end{pmatrix} \qquad P_1 = \begin{pmatrix} -B & 0 \\ 0 & \Gamma \end{pmatrix} \qquad P_2 = \begin{pmatrix} T & R \\ -R^{\mathrm{T}} & 0 \end{pmatrix} \tag{2.3}$$

where B, C, Γ, T and R are 3×3 matrices:

$$
\begin{aligned}
&B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & c\omega_3 & -\omega_2 \\ -c\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \\
&\Gamma = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix} \\
&T = \begin{pmatrix} 0 & -c\omega_3 & \omega_2/c \\ c\omega_3 & 0 & -\omega_1/c \\ -\omega_2/c & \omega_1/c & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2/c & \gamma_1/c & 0 \end{pmatrix}.
\end{aligned}
\tag{2.4}
$$

The Hamiltonian functions h_α can be written as

$$
\begin{aligned}
&h_0 = \tfrac{1}{2}F_4 + 2\sigma c F_1 F_3 \qquad h_1 = \sigma c^2 F_1^3 - F_3 - 2\sigma c F_1 F_2 \\
&h_2 = F_2
\end{aligned}
\tag{2.5}
$$

where $\sigma = (c-1)/2c$ and

$$
\begin{aligned}
&F_1 = \omega_3 && F_2 = \tfrac{1}{2}(\omega_1^2 + \omega_2^2 + c\omega_3^2) - \gamma_3 \\
&F_3 = \omega_1\gamma_1 + \omega_2\gamma_2 + c\omega_3\gamma_3 && F_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2.
\end{aligned}
\tag{2.6}
$$

As is known, the functions F_i $(i = 1, \ldots, 4)$ are integrals of motion for equation (2.1); they are independent and in involution w.r.t. each one of the three Poisson tensors. Moreover, (F_1, F_2) are Casimir functions of P_0, (F_1, F_4) of P_1 and (F_3, F_4) of P_2.

The vector field X_L can be immersed in two different bH chains, starting and ending with the Casimir functions of the Poisson tensors P_α:

$$
\begin{aligned}
&P_0\,\mathrm{d}F_2 = 0 \qquad P_2\,\mathrm{d}F_2 = P_0\,\mathrm{d}h_0 = X_L \\
&P_2\,\mathrm{d}h_0 = P_0\,\mathrm{d}(-\sigma F_3^2) \qquad P_2\,\mathrm{d}(-\sigma F_3^2) = 0; \\
&P_0\,\mathrm{d}F_2 = 0 \qquad P_1\,\mathrm{d}F_2 = P_0\,\mathrm{d}h_1 \qquad P_1\,\mathrm{d}h_1 = P_0\,\mathrm{d}h_0 = X_L \\
&P_1\,\mathrm{d}h_0 = P_0\,\mathrm{d}(-\sigma c F_1 F_4) \qquad P_1\,\mathrm{d}(-\sigma c F_1 F_4) = 0.
\end{aligned}
\tag{2.7}
$$

Remark 2.1. The Hamiltonian formulation of LT w.r.t. P_2 is classical (see, e.g., [12]). The bH formulation w.r.t. (P_0, P_2) was introduced in [13] in the semidirect product $\mathfrak{so}(3) \times \mathfrak{so}(3)$, and was later recovered in [6] in an algebraic–geometric setting. The tri-Hamiltonian formulation w.r.t. (P_0, P_1, P_2) was constructed in [14], by a suitable reduction of the Lie–Poisson pencil defined in the direct sum of three copies of $\mathfrak{so}(3)$. (To compare the above-quoted results, let us recall that the angular momentum and the vertical unit vector are taken as dynamical variables in [12–14], whereas the angular momentum is replaced by the angular velocity ω in [6] and in the present paper.)

3. The quasi-bi-Hamiltonian model

The qbH model was introduced in [2, 15] and developed in [3, 16] (see also [4] and references therein). Here we summarize some facts to be used in the rest of the paper.

Let Q_0, Q_1 be two compatible Poisson tensors on a manifold M; a vector field X is said to admit a qbH formulation w.r.t. Q_0 and Q_1 if there are three functions ρ, H, K such that

$$
X = Q_0\,\mathrm{d}H = \frac{1}{\rho} Q_1\,\mathrm{d}K. \tag{3.1}
$$

In other words, X is Hamiltonian w.r.t. Q_0 with Hamiltonian function H, and it is *quasi-Hamiltonian* (qH) w.r.t. Q_1, with qH function K and conformal factor $1/\rho$. In spite of the presence of ρ, equation (3.1) implies that H and K are in involution w.r.t. both Poisson brackets corresponding to Q_0 and Q_1 (as well as in the particular bH case $\rho = 1$).

If $\dim M = 2n$, the qbH formulation is said to be of maximal rank if at each point $m \in M$ the Poisson tensors Q_0, Q_1 are non-degenerate and the associated tensor $N = Q_1 Q_0^{-1}$ (with vanishing Nijenhuis torsion) has n independent eigenvalues $\lambda_1(m), \ldots, \lambda_n(m)$. In this case, one can introduce a local chart (λ_i, μ_i) $(i = 1, 2, \ldots, n)$, called a Darboux–Nijenhuis chart [17], such that Q_0, Q_1 and N take the canonical form

$$Q_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} \qquad N = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{3.2}$$

with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$; in general, the coordinate functions μ_i, canonically conjugate to λ_i, can be computed by quadratures. Finally, the qbH formulation is said to be of Pfaffian type if $\rho = \prod_{i=1}^n \lambda_i$.

The following result has been proved in [3] for a Pfaffian qbH vector field.

Proposition 3.1. *The general solution of equation (3.1) for the Pfaffian case is given by functions H and K which, in a Darboux–Nijenhuis chart (λ_i, μ_i), take the 'canonical' form*

$$H = \sum_{i=1}^n \frac{f_i}{\Delta_i} \qquad K = \sum_{i=1}^n \frac{\rho}{\lambda_i}\frac{f_i}{\Delta_i} \qquad \Delta_i = \prod_{j\neq i}(\lambda_i - \lambda_j) \tag{3.3}$$

where each f_i is an arbitrary function, depending at most on the pair (λ_i, μ_i). Moreover, the Hamilton–Jacobi equations for both H and K are separable.

This proposition has a straightforward consequence.

Corollary 3.2. *Let $X = Q_0\, \mathrm{d}H$ be a Hamiltonian vector field; if in a Q_0-Darboux chart (x, y) the Hamiltonian H takes the canonical form (3.3), then X admits a Pfaffian qbH formulation w.r.t. a Poisson tensor Q_1 and a qH function K of the form (3.2) and (3.3), respectively.*

Vice versa, let $X = (1/\rho)Q_1\, \mathrm{d}K$ be a qH vector field w.r.t. Q_1; if, in a chart (x, y), Q_1 and K take the canonical forms (3.2), (3.3) and $\rho = \prod_{i=1}^n x_i$, then also $X = Q_0\, \mathrm{d}H$ with Q_0 and H given by (3.2), (3.3), respectively. Hence, the chart (x, y) is a Darboux–Nijenhuis chart for the Poisson pair Q_0, Q_1.

For $n = 2$, this corollary can be slightly generalized, in a way that is useful for subsequent applications to LT.

Proposition 3.3. *Let S be a four-dimensional manifold and $Y = Q_0\, \mathrm{d}H$ be a Hamiltonian vector field w.r.t. a non-degenerate Poisson tensor Q_0. Let there be a Darboux chart (x, y) such that the Hamiltonian H can be written as a linear combination of two functions $\hat{H}$, $\hat{K}$ with the canonical form (3.3), i.e.,*

$$\begin{aligned} H(x, y) &= \beta \hat{H}(x, y) + \hat{K}(x, y) \qquad \beta = \mathrm{const} \\ \hat{H}(x, y) &= \frac{1}{x_1 - x_2}(\hat{f}_1(x_1, y_1) - \hat{f}_2(x_2, y_2)) \\ \hat{K}(x, y) &= \frac{1}{x_1 - x_2}(x_2 \hat{f}_1(x_1, y_1) - x_1 \hat{f}_2(x_2, y_2)). \end{aligned} \tag{3.4}$$

Then, the vector field Y admits the Pfaffian qbH formulation (3.1)–(3.3); a Darboux–Nijenhuis chart (λ, μ) is given by the following map:

$$\Phi : (x, y) \mapsto (\lambda, \mu) \qquad \lambda_i = \frac{1}{x_i + \beta} \qquad \mu_i = -y_i(x_i + \beta)^2 \qquad (i = 1, 2). \tag{3.5}$$

Hence, H is separable in the chart (λ, μ). Moreover, H is separable also in the chart (x, y) and the corresponding Hamilton–Jacobi equation

$$H(x_1, x_2, \partial W/\partial x_1, \partial W/\partial x_2) = h \tag{3.6}$$

has the complete solution $W(x_1, x_2; \hat{h}, \hat{k}) = W_1(x_1; \hat{h}, \hat{k}) + W_2(x_2; \hat{h}, \hat{k})$*,* W_1 *and* W_2 *fulfilling the Sklyanin separation equations [1]*

$$\hat{f}_1(x_1, W_1'(x_1)) = x_1\hat{h} - \hat{k} \qquad \hat{f}_2(x_2, W_2'(x_2)) = x_2\hat{h} - \hat{k} \tag{3.7}$$

with $\beta\hat{h} + \hat{k} = h$*.*

Proof. It is straightforward to check that the map $\Phi : (x, y) \mapsto (\lambda, \mu)$ is a Darboux map for Q_0; moreover, since $x_1 - x_2 = -(\lambda_1 - \lambda_2)/\lambda_1\lambda_2$, the Hamiltonian H takes the canonical form (3.3)

$$\begin{aligned} H(x(\lambda, \mu), y(\lambda, \mu)) &= \beta\hat{H}(x(\lambda, \mu), y(\lambda, \mu)) + \hat{K}(x(\lambda, \mu), y(\lambda, \mu)) \\ &= \frac{1}{\lambda_1 - \lambda_2}\left(-\lambda_1\hat{f}_1\left(\frac{1}{\lambda_1} - \beta, -\lambda_1^2\mu_1\right) + \lambda_2\hat{f}_2\left(\frac{1}{\lambda_2} - \beta, -\lambda_2^2\mu_2\right)\right) \\ &= \frac{1}{\lambda_1 - \lambda_2}(f_1(\lambda_1, \mu_1) - f_2(\lambda_2, \mu_2)) \end{aligned} \tag{3.8}$$

where

$$f_1(\lambda_1, \mu_1) = -\lambda_1\hat{f}_1\left(\frac{1}{\lambda_1} - \beta, -\lambda_1^2\mu_1\right) \qquad f_2(\lambda_2, \mu_2) = -\lambda_2\hat{f}_2\left(\frac{1}{\lambda_2} - \beta, -\lambda_2^2\mu_2\right). \tag{3.9}$$

On account of corollary 3.2, the vector field $Y = Q_0\,\mathrm{d}H$ admits the qH formulation $Y = (1/\rho)Q_1\,\mathrm{d}K$ and H is separable.

Obviously enough, H is separable also in the chart (x, y), since the map Φ is a *separated* map [18], i.e., it maps separated coordinates into separated ones. Indeed, taking into account the form (3.4) of the function H, it is easily checked that the Hamilton–Jacobi equation $H(x, \partial W/\partial x) = h$ has a complete solution $W(x_1, x_2; \hat{h}, \hat{k}) = W_1(x_1; \hat{h}, \hat{k}) + W_2(x_2; \hat{h}, \hat{k})$, with $\beta\hat{h} + \hat{k} = h$, and that W_1, W_2 fulfil the Sklyanin separation equations (3.7) for the Hamilton–Jacobi equations $\hat{H}(x, \partial W/\partial x) = \hat{h}$, $\hat{K}(x, \partial W/\partial x) = \hat{k}$. □

4. The reduction of the tri-Hamiltonian structure of the Lagrange top

If a vector field X on a manifold M is bH w.r.t. a pair of degenerate Poisson tensors (P_0, P_1), a preliminary step in analysing its integrability is trying to reduce the vector field, its Hamiltonian functions and the Poisson tensors on a lower-dimensional manifold M', where one of the two Poisson tensors, say P_0, is invertible. A natural way to do that is to fix the values of the Casimir functions of P_0. Of course, both P_0 and X can be properly *restricted* to a symplectic leaf S_0, giving rise to a Poisson tensor P_0' and to a vector field $X' = P_0'\,\mathrm{d}H'$, H' being the restriction to S_0 of the original Hamiltonian H. However, without additional assumptions, P_1 is not certain to restrict to S_0, so X' loses the original bH formulation.

This situation occurs also for the tri-Hamiltonian structure of the LT. Each one of the three Poisson tensors P_α has two independent Casimir functions, and the generic symplectic leaves S_α are four-dimensional submanifolds of M. On account of equation (2.6), they are defined as

$$\begin{aligned} S_0 &= \left\{m \in M | \omega_3 = a_1/2c, \omega_1^2 + \omega_2^2 + c\omega_3^2 - 2\gamma_3 = 2a_2\right\} \\ S_1 &= \left\{m \in M | \omega_3 = a_1/2c, \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = a_4\right\} \\ S_2 &= \{m \in M | \omega_1\gamma_1 + \omega_2\gamma_2 + c\omega_3\gamma_3 = -\tfrac{1}{2}a_3, \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = a_4\} \end{aligned} \tag{4.1}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants. Each Poisson tensor P_α can be properly restricted to a corresponding symplectic leaf S_α, but the other two tensors do not restrict to the same leaf.

Nevertheless, a quite general reduction technique given by the Marsden–Ratiu theorem [19] can be applied; it will enable us to construct on S_α a Poisson–Nijenhuis structure [9, 10] induced by the tri-Hamiltonian structure on M, and on S_0 a qbH formulation for the vector field X_L'. Essentially, one considers a Poisson manifold (M, P), a submanifold $S \hookrightarrow M$ and a distribution $D \subset TM_{|S}$ such that $E := D \cap TS$ is a regular foliation with a good quotient $\mathcal{N} = S/E$. Then, the theorem states that the Poisson tensor P is reducible to $\mathcal{N}$ if the following conditions hold:

(i) the functions on M which are invariant along D form a Poisson subalgebra of $C^\infty(M)$;
(ii) $P(D^0) \subset TS + D$ (D^0 being the annihilator of D in T^*M).

Analogously to previous applications of this procedure to bH structures [5, 8, 20], let us choose as the submanifold S a generic symplectic leaf S_α of the Poisson tensor P_α and a distribution D such that at each point $s_\alpha \in S_\alpha$ the following decomposition holds:

$$T_{s_\alpha} M = T_{s_\alpha} S_\beta \oplus D_{s_\alpha} \tag{4.2}$$

where S_β is the symplectic leaf of P_β ($\beta = 0, 1, 2$) passing through s_α.

This assumption ensures that (ii) is trivially fulfilled and that $E = 0$, so the reduction procedure becomes a *submersion* $\Pi : M \to S_\alpha$ onto the manifold S_α; thus, it allows us to endow S_α with a non-degenerate tri-Hamiltonian structure, since the kernels of the reduced Poisson tensors P_β' vanish. Indeed, if Π^* denotes the (injective) pull-back of the submersion Π, we have

$$\mathrm{Ker}_{s_\alpha} P_\beta' = (\Pi^*)^{-1}(\mathrm{Im}_{s_\alpha}\Pi^* \cap P_\beta^{-1}(D_{s_\alpha} \cap T_{s_\alpha} S_\beta)) \overset{(4.2)}{=} (\Pi^*)^{-1}(\mathrm{Im}_{s_\alpha}\Pi^* \cap \mathrm{Ker}_{s_\alpha} P_\beta) = 0 \tag{4.3}$$

where we have taken into account that

$$\mathrm{Im}_{s_\alpha}\Pi^* \subset D^0 \qquad D^0 \cap \mathrm{Ker}_{s_\alpha} P_\beta = D^0 \cap (\mathrm{Im}_{s_\alpha} P_\beta)^0 = D^0 \cap (T_{s_\alpha} S_\beta)^0 \overset{(4.2)}{=} 0. \tag{4.4}$$

In the LT case, the distribution is as follows.

Lemma 4.1. *Let D be the distribution given by the vector fields*

$$Z_1 = -\mathrm{i}c\frac{\partial}{\partial\omega_2} + \frac{\partial}{\partial\omega_3} \qquad Z_2 = \mathrm{i}\frac{\partial}{\partial\gamma_2} - \frac{\partial}{\partial\gamma_3} \tag{4.5}$$

($\mathrm{i} = \sqrt{-1}$). Moreover, let φ_1, φ_2 be two generic functions. Then, for each Poisson tensor P_α there are two vector fields $W_{1\alpha}$ and $W_{2\alpha}$ (depending on φ_1 and φ_2) such that

$$L_{\varphi_1 Z_1 + \varphi_2 Z_2}(P_\alpha) = Z_1 \wedge W_{1\alpha} + Z_2 \wedge W_{2\alpha} \tag{4.6}$$

(L_Z and $\wedge$ denoting the Lie derivative along the flow of the vector field Z and the exterior product of vector fields, respectively).

Proof. It is easy to check that $L_{Z_j} P_\alpha = Z_1 \wedge Y_{1j\alpha} + Z_2 \wedge Y_{2j\alpha}$ ($j = 1, 2$), with suitable vector fields $Y_{1j\alpha}, Y_{2j\alpha}$. This result, together with the identity $L_{fX}(P) = f L_X(P) + X \wedge P\,\mathrm{d}f$, implies (4.6), the vector fields $W_{j\alpha}$ being $W_{j\alpha} = \varphi_1 Y_{j1\alpha} + \varphi_2 Y_{j2\alpha} + P_\alpha\,\mathrm{d}\varphi_j$. □

Equation (4.6) implies the assumption (i), since if f and g are invariant functions along D and $Z \in D$, then $L_{\varphi Z}\{f, g\} = \langle \mathrm{d}f, L_{\varphi Z}(P)\,\mathrm{d}g\rangle \overset{(4.6)}{=} 0$ for each function φ. Moreover, condition (4.2) is generically satisfied, as can be easily verified. Hence, conditions (i), (ii) are fulfilled and the Marsden–Ratiu reduction technique can be applied on each symplectic leaf S_α. In conclusion, we have proved the following.

Proposition 4.2. *The tri-Hamiltonian structure P_β is reducible to a non-degenerate tri-Hamiltonian structure P_β' on each one of the symplectic leaves S_α.*

To express the reduced tensors in a particularly simple and useful form, it is convenient to adapt the coordinates on M to the distribution D, introducing a parametrization including coordinate functions which span the subalgebra of the functions invariant along D. Let us choose the chart (u, v, w), related to (ω, γ) by the map $\Psi : M \to M : (\omega, \gamma) \mapsto (u, v, w)$:

$$\begin{array}{llll} u_1 = c\omega_3 - \mathrm{i}\omega_2 & u_2 = \mathrm{i}\gamma_2 - \gamma_3 & & \\ v_1 = \omega_1 & v_2 = -\gamma_1 & w_1 = \mathrm{i}\omega_2 + c\omega_3 & w_2 = -\mathrm{i}\gamma_2 - \gamma_3. \end{array} \tag{4.7}$$

Taking into account the tri-Hamiltonian structure P_α given by (2.3) and the definition (4.1) of S_α, a straightforward (though lengthy) calculation allows one to verify that the chart (u, v) gives a parametrization on each one of the symplectic leaves S_α; the reduced Poisson tensors P'_β and the tensor N take the form

$$P'_0 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & u_1 \\ 0 & -1 & 0 & 0 \\ -1 & -u_1 & 0 & 0 \end{pmatrix} \qquad P'_1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -u_2 \\ -1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \end{pmatrix}$$
$$P'_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & -u_1 & -u_2 \\ 0 & 0 & -u_2 & 0 \\ u_1 & u_2 & 0 & 0 \\ u_2 & 0 & 0 & 0 \end{pmatrix}. \tag{4.8}$$

Remark 4.3. By a direct inspection, one easily concludes that the tensor $N' := P'_1 P'^{-1}_0$ (with vanishing Nijenhuis torsion) is such that $P'_1 = N' P'_0$ and $P'_2 = N' P'_1$.

The matrix representation of P'_0 and of the adjoint tensor N'^* of N' are formed by Hankel and Frobenius blocks, respectively, so (u, v) are Hankel–Frobenius coordinates, in the terminology of [8].

Proposition 4.4. *Let us consider the map* $\Psi : S_\alpha \to S_\alpha : (u, v) \mapsto (x, y)$:

$$\begin{array}{ll} x_1 = \frac{1}{2}\left(-u_1 - \sqrt{u_1^2 - 4u_2}\right) & x_2 = \frac{1}{2}\left(-u_1 + \sqrt{u_1^2 - 4u_2}\right) \\ y_1 = \frac{1}{2}\left(2v_2 - u_1 v_1 - v_1\sqrt{u_1^2 - 4u_2}\right) & y_2 = \frac{1}{2}\left(2v_2 - u_1 v_1 + v_1\sqrt{u_1^2 - 4u_2}\right). \end{array} \tag{4.9}$$

The chart (x, y) *is a Darboux–Nijenhuis chart for the tri-Hamiltonian structure on* S_α*; the reduced Poisson tensors* P'_α *have the block-matrix forms*

$$P'_0 = -\mathrm{i}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad P'_1 = -\mathrm{i}\begin{pmatrix} 0 & \mathcal{X} \\ -\mathcal{X} & 0 \end{pmatrix} \qquad P'_2 = -\mathrm{i}\begin{pmatrix} 0 & \mathcal{X}^2 \\ -\mathcal{X}^2 & 0 \end{pmatrix} \tag{4.10}$$

where $\mathcal{X} = \mathrm{diag}(x_1, x_2)$.

Proof. A straightforward computation, taking into account equations (4.8) and (4.9). $\square$

(To be more precise, in order to have the Darboux–Nijenhuis chart defined in section 3, one should eliminate the factor $(-\mathrm{i})$ in equation (4.10), via the map $x \mapsto \mathrm{i}x$, $y \mapsto y$.)

5. The reduction of the vector field and the Hamiltonians of the Lagrange top

Having established the projection of the tri-Hamiltonian structure on each one of the symplectic leaves S_α, the next step is to consider the reduction of the vector field X_L and of the corresponding Hamiltonian functions h_α.

Unfortunately, they *do not project* onto S_α, since X_L does not preserve the distribution D and the Hamiltonians h_α are not invariant along D; hence, the tri-Hamiltonian formulation of X_L is lost on S_α. Nevertheless, each pair (X_L, h_α) can be *restricted* to the corresponding symplectic leaf S_α, so that equation (2.1), restricted to S_α, keeps a Hamiltonian formulation. Furthermore, if we consider the reduction on a symplectic leaf S_0, we can recover, as a reminder of the original tri-Hamiltonian formulation, a qbH formulation for X_L; this suffices to provide a set of separation variables. Indeed, the following holds.

Proposition 5.1. *The vector field X_L, restricted to S_0, takes the form*

$$X_L = P_0' \,\mathrm{d}H = -\mathrm{i}Q_0 \,\mathrm{d}H. \tag{5.1}$$

Its Hamiltonian $H = h_{0|S_0}$ takes the form

$$H(x, y) = \sigma a_1 \hat{H}(x, y) + \hat{K}(x, y) \tag{5.2}$$

where

$$\begin{aligned}
\hat{H}(x, y) &= \frac{1}{x_1 - x_2}(\hat{f}(x_1, y_1) - \hat{f}(x_2, y_2)) \\
\hat{K}(x, y) &= \frac{1}{x_1 - x_2}(x_2 \hat{f}(x_1, y_1) - x_1 \hat{f}(x_2, y_2)) \\
\hat{f}(\xi, \eta) &= -\frac{1}{2}\eta^2 + \frac{1}{2}\xi^4 + \frac{1}{2}a_1\xi^3 + \left(a_2 + \sigma\frac{a_1^2}{4}\right)\xi^2.
\end{aligned} \tag{5.3}$$

Proof. A straightforward computation. □

On account of this result, we are just in the situation considered in proposition 3.3, with

$$\beta = \sigma a_1 \qquad \hat{f}_1 = \hat{f}_2 = \hat{f}. \tag{5.4}$$

So, X_L admits a qbH formulation; the Darboux–Nijenhuis coordinates (λ, μ) are obtained from (x, y) via the map (3.5):

$$\lambda_i = (x_i + \sigma a_1)^{-1} \qquad \mu_i = -y_i\,(x_i + \sigma a_1)^2 \qquad (i = 1, 2). \tag{5.5}$$

As follows from the general results of propositions 3.1, 3.3, H and K are separable both in the Darboux–Nijenhuis chart (λ, μ) and in the chart (x, y). Using the latter, let us compute a solution W of the Hamilton–Jacobi equations for H and K:

$$H\left(x_1, x_2, \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}\right) = h \qquad K\left(x_1, x_2, \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}\right) = k. \tag{5.6}$$

Taking into account the expression (5.3) of $\hat{f}$ and the fact that the qH function K given by (3.3) turns out to be $K = \hat{H}$, we have

$$\begin{aligned}
W(x_1, x_2; h, k) &= \int^{x_1} \sqrt{g(\xi)}\,\mathrm{d}\xi + \int^{x_2} \sqrt{g(\xi)}\,\mathrm{d}\xi \\
g(\xi) &= \xi^4 + a_1\xi^3 + \left(2a_2 + \sigma\frac{a_1^4}{2}\right)\xi^2 - 2k\xi + 2(h - \sigma a_1 k).
\end{aligned} \tag{5.7}$$

6. Concluding remarks

The first result in this paper is that, reducing *à la Marsden–Ratiu* the tri-Hamiltonian structure (P_0, P_1, P_2) of LT onto a generic symplectic leaf S_α of each Poisson tensor, a non-degenerate Poisson–Nijenhuis structure is obtained. The reduction depends essentially on the distribution D fulfilling (4.2) and (4.6); since D may be not unique, possibly different Poisson–Nijenhuis structures can be constructed on the symplectic leaf. This point deserves further investigation.

The second step of the reduction procedure is the restriction of the LT vector field and Hamiltonian functions to the invariant submanifold S_0, discussed in section 5. This produces a qbH formulation for the LT vector field and consequently, as a necessary outcome, a set of separation variables. An open question is whether the restriction of the LT vector field to other invariant submanifolds, such as the symplectic leaves S_1 and S_2 of the Poisson tensors P_1 and P_2, gives rise to different sets of separation variables.

As a last remark, we observe that the tri-Hamiltonian structure of LT has a deformation in the original phase space M (see, e.g., [14]). In fact, there is a vector field τ such that $L_\tau(P_2) = 2P_1$, $L_\tau(P_1) = P_0$, $L_\tau(P_0) = 0$; in the chart (ω, γ) chosen in this paper, τ is given by $\tau = (0, 0, -2/c, \omega_1, \omega_2, c\omega_3)^{\mathrm{T}}$. In contrast, a recursion operator N relating the Poisson tensors does not exist in M. Under the submersion $\Pi : M \to S_0$, the deformation process is preserved since the vector field τ is projectable onto S_0; hence, the previous relations hold for (P_0', P_1', P_2') w.r.t. the projected vector field τ', given by $\tau' = -(2, u_1, 0, v_1,)^{\mathrm{T}}$ in the chart (u, v). As observed in remark 4.3, on S_0 there is also an invertible recursion operator N' such that $P_1' = N'P_0'$ and $P_2' = N'P_1'$ (consequently, one has a whole sequence of compatible Poisson tensors $P_{j+1}' = N'P_j'$ for each integer j). One may wonder whether the recursion scheme based on N' could be inferred from the existence of the deformation scheme on the initial phase space, and under which conditions on the deformation vector field τ. To the best of our knowledge, this question (which is not specific to the LT) has not yet received a satisfactory answer; in our opinion, it deserves further investigation in the general framework of the reduction theory for multi-Hamiltonian manifolds.

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